Math 246B Lecture 12 Notes

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February 4, 2019

1 Applications of Runge's Theorem

1.1 Locally uniform approximation of holomorphic functions

Last time, we showed that if $\Omega \subseteq \mathbb{C}$, we can find an increasing sequence $K_n \subseteq \Omega$ of compact sets such that $\Omega = \bigcup_{n=1}^\infty K_n$ and such that every bounded component of $\mathbb{C} \setminus K_n$ contains a bounded component contains a bounded component of $\mathbb{C} \setminus \Omega$.

Corollary 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \mathbb{C} \setminus \Omega$ be such that each bounded component of $\mathbb{C} \setminus \Omega$ meets A. Let $f \in \mathrm{Hol}(\Omega)$. Then there exist rational functions r_n that have no poles outside of A such that $r_n \to f$ locally uniformly in Ω. If $\mathbb{C} \setminus \Omega$ has no bounded component,, then there exists a sequence of polynomials p_n such that $p_n \to f$ locally uniformly in Ω.*

Proof. Let (K_n) be a compact exhaustion as before. By Runge's theorem and the property of the compact exhaustion, for every n, there exists a rational function r_n with no poles outside of A such that $|f - r_n| \leq 1/n$ on K_n. Since any compact $K \subseteq K_N \subseteq K_n$ for large $n \geq N$, we get $r_n \to f$ uniformly on K.

If $\mathbb{C} \setminus \Omega$ has no bounded component, then none of the sets $\mathbb{C} \setminus K_n$ has a bounded component. By Runge's theorem, for any n, there is a polynomial p_n such that $|f - p_n| \leq 1/n$ on K_n. So $p_n \to f$ locally uniformly in Ω. □

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of $\mathbb{C}$.

Corollary 1.2. *Let $\Omega \subseteq \mathbb{C}$ be such that $\hat{\mathbb{C}} \setminus \Omega$ is connected. Let $f \in \mathrm{Hol}(\Omega)$. Then there exist polynomials p_n such that $p_n \to f$ locally uniformly.*

Proof. If suffices to show that $\mathbb{C} \setminus K_n$ has no bounded component for all n. For contradiction, let V be a bounded component of $\mathbb{C} \setminus K_n$. Then there is a bounded component C of $\mathbb{C} \setminus \Omega$ such that $C \subseteq V$. In particular, $(\hat{\mathbb{C}} \setminus \Omega) \cap V \neq \varnothing$. Let $V' \subseteq \hat{\mathbb{C}}$ be the union of all the other components of $\mathbb{C} \setminus K_n$ (including the unbounded one) and $\{\infty\}$. Then $V \cap V' = \varnothing$, V and V' are open in $\hat{\mathbb{C}}$, and $V \cup V' \supseteq \hat{\mathbb{C}} \setminus \Omega$: $(\hat{\mathbb{C}} \setminus \Omega) \cap V \neq \varnothing$, and $(\hat{\mathbb{C}} \setminus \Omega) \cap V' \neq \varnothing$ (because ∞ is in the intersection). This contradicts the assumption that $\hat{\mathbb{C}} \setminus K_n$ is connected. □

1.2 Solving the inhomogeneous Cauchy-Riemann equation

Earlier, we solved the inhomogeneous Cauchy-Riemann equation for functions which are compactly supported. We even had a formula for it. Let's show a related result for non-compactly supported funcitons.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in C^1(\Omega)$. Then there exists $u \in C^1(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = f$ in Ω.*

Proof. Let $(K_j)_{j\geq 1}$ be a compact exhaustion of Ω, as before. Let $\psi_j \in C_0^1(\Omega)$ be such that $0 \leq \psi_j \leq 1$ and $\psi_j = 1$ near K_j. Let

$$\varphi_j = \begin{cases} \psi_j - \psi_{j-1} & j > 1 \\ \psi_j & j = 1. \end{cases}$$

Then $\varphi_j \in C_0^1(\Omega)$, $\varphi_j = 0$ in a neighborhood of K_{j-1}, and sum $\sum_{j=1}^\infty \varphi_j$ has only finitely many nonzero terms for each $x \in \Omega$ (and hence converges). We can calculate

$$\sum_{j=1}^\infty \varphi_j = \lim_{N\to\infty} \sum_{j=1}^N \varphi_j = \lim_{N\to\infty} (\psi_1 + \sum_{j=2}^N (\psi_j - \psi_{j-1})) = \lim_{N\to\infty} (\psi_1 + \psi_N - \psi_1) = 1.$$

This is called a **locally finite paritition of unity**. Write $f = \sum_{j=1}^\infty \varphi_j f$, where $\varphi_j f \in C_0^1(\Omega) \subseteq C_0^1(\mathbb{C})$. As $u_j f$ is compactly supported, there exists a function $u_j \in C^1(\mathbb{C})$ such that $\frac{\partial u_j}{\partial \bar{z}} = \varphi_j f$ (we can take $u_j(z) = (1/\pi) \iint \varphi_j f(\zeta)/(z-\zeta)\, L(ds))$.

Here is the problem: the sum $\sum_j u_j$ may not converge. We know that $\frac{\partial u_j}{\partial \bar{z}} = 0$ in a neighborhood of K_{j-1}, so u_j is holomorphic near K_{j-1}. By Runge's theorem, there exists a function $v_j \in \mathrm{Hol}(\Omega)$ such that $|u_j - v_j| \leq 2^{-j}$ on K_{j-1} for all j. Now try the sum $u = \sum_{j=1}^\infty (u_j - v_j)$. We claim that $u \in C^1(\Omega)$ and $\frac{\partial u}{\partial \bar{z}} = f$. Let $K \subseteq \Omega$ be compact, and let N be such that $K \subseteq K_N$. Then

$$u = \sum_{j=1}^N (u_j - v_j) + \sum_{j=N+1}^\infty (u_j - v_j),$$

and $|u_j - v_j| \leq 2^{-j}$ on K, so $u \in C(\Omega)$. Since $\partial_{\bar{z}}(u_j - v_j) = 0$ in a neighborhood of K_{j-1}, $u_j - v_j$ is holomorphic in a neighborhood of K_N, where $j \geq N+1$. So the sum of the series $\sum_{j=N+1}^\infty (u_j - v_j)$ is holomorphic in K_N. Thus, $u \in C^1(\Omega)$, and we compute in K_N^o:

$$\frac{\partial}{\partial \bar{z}} = \sum_{j=1}^N \partial_{z_j}(u_j - v_j) = \sum_{j=1}^N \varphi_j f = \left(\sum_{j=1}^N \varphi_j + \underbrace{\sum_{j=N+1}^\infty \varphi_j}_{=0 \text{ in } K_N} \right) f = f.$$

□